Gravitational waves in vacuum spacetimes with cosmological constant. II. Deviation of geodesics and interpretation of non-twisting type N solutions

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Abstract

In a suitably chosen essentially unique frame tied to a given observer in a general spacetime, the equation of geodesic deviation can be decomposed into a sum of terms describing specific effects: isotropic (background) motions associated with the cosmological constant, transverse motions corresponding to the effects of gravitational waves, longitudinal motions, and Coulomb-type effects. Conditions under which the frame is parallelly transported along a geodesic are discussed. Suitable coordinates are introduced and an explicit coordinate form of the frame is determined for spacetimes admitting a non-twisting null congruence. Specific properties of all non-twisting type N vacuum solutions with cosmological constant Λ (non-expanding Kundt class and expanding Robinson-Trautman class) are then analyzed. It is demonstrated that these spacetimes can be understood as exact transverse gravitational waves of two polarization modes "+" and "×", shifted by $\frac{\pi}{4}$, which propagate "on" Minkowski, de Sitter, or anti-de Sitter backgrounds. It is also shown that the solutions with $\Lambda > 0$ may serve as exact demonstrations of the cosmic "no-hair" conjecture in radiative spacetimes with no symmetry.

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I. INTRODUCTION AND SUMMARY

In the preceding paper [1] we classified non-twisting type N solutions of the vacuum Einstein's equations with a non-vanishing cosmological constant Λ and analyzed their geometrical properties. Here we wish to discuss their physical properties. We shall show that these solutions can be interpreted as gravitational waves propagating in spacetimes of constant curvature — in Minkowski, de Sitter, or anti-de Sitter spaces. Our treatment focuses on the analysis of the equation of geodesic deviation.

We first discuss the equation of geodesic deviation in general spacetimes (Section II), briefly reviewing and extending [2]-[4] by using both a Newman-Penrose null tetrad and a physical frame of four independent vectors $\{\mathbf{e}_{(a)}\}$ tied to the geodesic with respect to which the relative motion is studied. In type N solutions only the Newman-Penrose scalar Ψ_4 is non-vanishing.

Starting from Section III we study non-twisting type N solutions with Λ. As shown in [1], they comprise the non-expanding Kundt class $KN(\Lambda)$ and the expanding Robinson-Trautman class $RTN(\Lambda, \epsilon)$. By analyzing the geodesic deviation in these spacetimes we demonstrate that they can be interpreted as exact transverse gravitational waves with two polarization modes (shifted by $\frac{\pi}{4}$) propagating "on" Minkowski, de Sitter, or anti-de Sitter space (depending on values of Λ). In the Appendix we calculate exact forms of wave amplitudes.

At the end of Section IV we discuss, for $\Lambda > 0$, special timelike geodesics explicitly. We demonstrate that observers moving along these geodesics see waves decaying exponentially fast and the spacetimes to approach locally the de Sitter space — in agreement with the cosmic no-hair conjecture (see e.g. [5], and references therein). As in our previous work [6], [7], this is an explicit demonstration of the conjecture under the presence of waves within exact theory.

II. THE RELATIVE MOTION OF FREE PARTICLES IN A GENERAL SPACETIME

It is natural to base the local characterization of radiative spacetimes on the equation of geodesic deviation [2]-[4]

$$\frac{D^2 Z^\mu}{d\tau^2} = -R^\mu{}_{\alpha\beta\gamma} u^\alpha Z^\beta u^\gamma \ , \tag{1}$$

where $\mathbf{u} = d\mathbf{x}/d\tau$, $\mathbf{u}\cdot\mathbf{u} = -1$, is the four-velocity of a free test particle (observer), τ is the proper time, and $\mathbf{Z}(\tau)$ is the displacement vector. In order to obtain invariant results one sets up a frame $\{\mathbf{e}_{(a)}\}$ along the geodesic. The frame components $Z^{(a)}(\tau)$, $\mathbf{Z} = Z^{(a)}\mathbf{e}_{(a)}$, are invariant quantities. Choosing $\mathbf{e}_{(0)} = \mathbf{u}$ and perpendicular space-like unit vectors $\{\mathbf{e}_{(1)}, \mathbf{e}_{(2)}, \mathbf{e}_{(3)}\}$ in the local hypersurface orthogonal to $\mathbf{u}$, we have $\mathbf{e}_{(a)}\cdot\mathbf{e}_{(b)} \equiv g_{\alpha\beta}e^{\alpha}_{(a)}e^{\beta}_{(b)} = \eta_{(a)(b)} = \mathrm{diag}(-1,1,1,1)$. The dual basis is $\mathbf{e}^{(0)} = -\mathbf{u}$ and $\mathbf{e}^{(i)} = \mathbf{e}_{(i)}$, $i = 1,2,3$. By projecting (1) onto the frame we get

$$\ddot{Z}^{(i)} = -R^{(i)}_{\ \ (0)(j)(0)}Z^{(j)} \ , \tag{2}$$

where $Z^{(j)} = \mathbf{e}^{(j)}\cdot\mathbf{Z} = e^{(j)}_{\mu}Z^{\mu}$ determine directly the distance between close test particles,

$$\ddot{Z}^{(i)} \equiv \mathbf{e}^{(i)}\cdot\frac{D^2\mathbf{Z}}{d\tau^2} = e^{(i)}_{\mu}\frac{D^2Z^{\mu}}{d\tau^2} \tag{3}$$

are physical relative accelerations, and $R_{(i)(0)(j)(0)} = e^{\alpha}_{(i)}u^{\beta}e^{\gamma}_{(j)}u^{\delta}R_{\alpha\beta\gamma\delta}$. Eq. (1) also implies $d^2Z^{(0)}/d\tau^2 = -u_{\mu}D^2Z^{\mu}/d\tau^2 = R_{\mu\alpha\beta\gamma}u^{\mu}u^{\alpha}Z^{\beta}u^{\gamma} = 0$ so that $Z^{(0)} = a_0\tau + b_0$, a_0, b_0 are constants. Setting $Z^{(0)} = 0$, all test particles are "synchronized" by τ (they always stay in the same local hypersurface). From the definition of the Weyl tensor we get $R_{(i)(0)(j)(0)} = C_{(i)(0)(j)(0)} + \frac{1}{2}(\delta_{ij}R_{(0)(0)} - R_{(i)(j)}) + \frac{1}{6}R\delta_{ij}$. Using Einstein's equations,

$$R_{(i)(0)(j)(0)} = C_{(i)(0)(j)(0)} - \frac{\Lambda}{3}\delta_{ij} - \frac{\kappa}{2}\left[T_{(i)(j)} - \delta_{ij}(T_{(0)(0)} + \frac{2}{3}T)\right] \ , \tag{4}$$

$T = T^{(a)}_{(a)}$. Following [8] we introduce null complex tetrad $\{\mathbf{e}_{\hat{a}}\} = \{\mathbf{e}_{\hat{1}}, \mathbf{e}_{\hat{2}}, \mathbf{e}_{\hat{3}}, \mathbf{e}_{\hat{4}}\} = \{\mathbf{m}, \bar{\mathbf{m}}, \mathbf{l}, \mathbf{k}\}$,

$$\begin{aligned} \mathbf{m} &= \tfrac{1}{\sqrt{2}}\left(\mathbf{e}_{(1)} + i\mathbf{e}_{(2)}\right), \ \bar{\mathbf{m}} = \tfrac{1}{\sqrt{2}}\left(\mathbf{e}_{(1)} - i\mathbf{e}_{(2)}\right) \ , \\ \mathbf{l} &= \tfrac{1}{\sqrt{2}}\left(\mathbf{u} - \mathbf{e}_{(3)}\right), \ \mathbf{k} = \tfrac{1}{\sqrt{2}}\left(\mathbf{u} + \mathbf{e}_{(3)}\right) \ . \end{aligned} \tag{5}$$

Null tetrad components of the Weyl tensor are (see e.g. [8], [9])

$$\begin{aligned} &\Psi_0 = C_{\alpha\beta\gamma\delta}k^{\alpha}m^{\beta}k^{\gamma}m^{\delta} \ , \qquad \Psi_1 = C_{\alpha\beta\gamma\delta}k^{\alpha}l^{\beta}k^{\gamma}m^{\delta} \ , \\ &\Psi_2 = \tfrac{1}{2}C_{\alpha\beta\gamma\delta}k^{\alpha}l^{\beta}(k^{\gamma}l^{\delta} - m^{\gamma}\bar{m}^{\delta}) \ , \\ &\Psi_3 = C_{\alpha\beta\gamma\delta}l^{\alpha}k^{\beta}l^{\gamma}\bar{m}^{\delta} \ , \qquad \Psi_4 = C_{\alpha\beta\gamma\delta}l^{\alpha}\bar{m}^{\beta}l^{\gamma}\bar{m}^{\delta} \ . \end{aligned} \tag{6}$$

Regarding expressions (6) and inverting relations (5) we obtain

$$\begin{aligned} &C_{(1)(0)(1)(0)} = \tfrac{1}{2}\mathcal{R}e\Psi_0 + \tfrac{1}{2}\mathcal{R}e\Psi_4 - \mathcal{R}e\Psi_2 \ , \quad C_{(2)(0)(2)(0)} = -\tfrac{1}{2}\mathcal{R}e\Psi_0 - \tfrac{1}{2}\mathcal{R}e\Psi_4 - \mathcal{R}e\Psi_2 \ , \\ &C_{(1)(0)(2)(0)} = \tfrac{1}{2}\mathcal{I}m\Psi_0 - \tfrac{1}{2}\mathcal{I}m\Psi_4 \ , \quad C_{(3)(0)(3)(0)} = 2\,\mathcal{R}e\Psi_2 \ , \\ &C_{(1)(0)(3)(0)} = -\mathcal{R}e\Psi_1 + \mathcal{R}e\Psi_3 \ , \quad C_{(2)(0)(3)(0)} = -\mathcal{I}m\Psi_1 - \mathcal{I}m\Psi_3 \ . \end{aligned} \tag{7}$$

Substituting Eqs. (4) and (7) into Eq. (2) we arrive at:

$$
\begin{aligned}
\ddot{Z}^{(1)} &= \frac{\Lambda}{3} Z^{(1)} - \frac{\kappa}{2}(T_{(0)(0)} + \frac{2}{3}T) Z^{(1)} + \frac{\kappa}{2} T_{(1)(j)} Z^{(j)} + \mathcal{G}_1 \ , \\
\ddot{Z}^{(2)} &= \frac{\Lambda}{3} Z^{(2)} - \frac{\kappa}{2}(T_{(0)(0)} + \frac{2}{3}T) Z^{(2)} + \frac{\kappa}{2} T_{(2)(j)} Z^{(j)} + \mathcal{G}_2 \ , \\
\ddot{Z}^{(3)} &= \frac{\Lambda}{3} Z^{(3)} - \frac{\kappa}{2}(T_{(0)(0)} + \frac{2}{3}T) Z^{(3)} + \frac{\kappa}{2} T_{(3)(j)} Z^{(j)} + \mathcal{G}_3 \ ,
\end{aligned}
\tag{8}
$$

where

$$
\begin{aligned}
\mathcal{G}_1 &\equiv +\mathcal{C} Z^{(1)} - (\mathcal{L}_1 - \mathcal{M}_1) Z^{(3)} - (\mathcal{A}_+ + \mathcal{B}_+) Z^{(1)} + (\mathcal{A}_\times - \mathcal{B}_\times) Z^{(2)} \ , \\
\mathcal{G}_2 &\equiv +\mathcal{C} Z^{(2)} + (\mathcal{L}_2 + \mathcal{M}_2) Z^{(3)} + (\mathcal{A}_+ + \mathcal{B}_+) Z^{(2)} + (\mathcal{A}_\times - \mathcal{B}_\times) Z^{(1)} \ , \\
\mathcal{G}_3 &\equiv -2\mathcal{C} Z^{(3)} - (\mathcal{L}_1 - \mathcal{M}_1) Z^{(1)} + (\mathcal{L}_2 + \mathcal{M}_2) Z^{(2)} \ ,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{C} = \mathcal{R}e\Psi_2 \ , \quad \mathcal{L}_1 = \mathcal{R}e\Psi_3 \ , \quad \mathcal{L}_2 = \mathcal{I}m\Psi_3 \ , \quad \mathcal{M}_1 = \mathcal{R}e\Psi_1 \ , \quad \mathcal{M}_2 = \mathcal{I}m\Psi_1 \ , \\
&\mathcal{A}_+ = \tfrac{1}{2}\mathcal{R}e\Psi_4 \ , \quad \mathcal{A}_\times = \tfrac{1}{2}\mathcal{I}m\Psi_4 \ , \quad \mathcal{B}_+ = \tfrac{1}{2}\mathcal{R}e\Psi_0 \ , \quad \mathcal{B}_\times = \tfrac{1}{2}\mathcal{I}m\Psi_0 \ .
\end{aligned}
\tag{9}
$$

Equations (8) are well suited for physical interpretation. The relative motions depend on: 1. the cosmological constant Λ responsible for overall background isotropic motions; 2. the energy-momentum tensor $T_{(a)(b)}$ terms describing interaction with matter-content; 3. the terms depending on the local free gravitational field, and consisting of Coulomb, longitudinal and transverse (outgoing/ingoing) components with amplitudes given by Ψ_A's. In the following we put $T_{(a)(b)} = 0$. Individual terms in Eq. (8) can be interpreted as follows:

Λ-term: Assuming $T_{(a)(b)} = 0 = \Psi_A$, Eq. (8) reduces to

$$\ddot{Z}^{(i)} = \frac{\Lambda}{3} Z^{(i)} \ . \tag{10}$$

Considering a sphere of test particles each having a position vector $\mathbf{Z}$, Eq. (10) implies that the acceleration of each particle is in the direction $\mathbf{Z}$ and has the same magnitude. Assume the frame $\{\mathbf{e}_{(i)}\}$ to be parallelly transported so that $\ddot{Z}^{(i)} = D^2(\mathbf{e}^{(i)} \cdot \mathbf{Z})/d\tau^2 = d^2 Z^{(i)}/d\tau^2$. Eqs. (10) have solutions

$$
\begin{aligned}
Z^{(i)}(\tau) &= A_i \, \tau + B_i && \text{for } \Lambda = 0 \ , \\
Z^{(i)}(\tau) &= A_i \exp(\sqrt{\Lambda/3}\,\tau) + B_i \exp(-\sqrt{\Lambda/3}\,\tau) && \text{for } \Lambda > 0 \ , \\
Z^{(i)}(\tau) &= A_i \cos(\sqrt{-\Lambda/3}\,\tau) + B_i \sin(\sqrt{-\Lambda/3}\,\tau) && \text{for } \Lambda < 0 \ ,
\end{aligned}
\tag{11}
$$

where A_i, B_i are constants. As expected, conformally flat ($\Psi_A = 0$) vacuum backgrounds (Minkowski, de Sitter or anti-de Sitter) are homogeneous and isotropic, so that the relative motion of test particles is isotropic.

Ψ_4-term: Assuming $\Lambda = 0 = T_{(a)(b)}$ and $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$, Eq. (8) reduces to

$$\begin{aligned} \ddot{Z}^{(1)} &= -\mathcal{A}_+ Z^{(1)} + \mathcal{A}_\times Z^{(2)} \ , \\ \ddot{Z}^{(2)} &= \quad \mathcal{A}_+ Z^{(2)} + \mathcal{A}_\times Z^{(1)} \ , \\ \ddot{Z}^{(3)} &= 0 \ , \end{aligned} \tag{12}$$

which describe the influence of "+" and "×" polarization modes of a transverse gravitational wave with amplitudes $\mathcal{A}_+$ and $\mathcal{A}_\times$. If particles, initially at rest, lie in the $(\mathbf{e}_{(1)}, \mathbf{e}_{(2)})$ plane, there is no motion in the longitudinal direction of $\mathbf{e}_{(3)}$. The ring of particles is deformed into an ellipse, the axes of different polarizations are shifted one with respect to the other by $\frac{\pi}{4}$ (such behavior is typical for linearized gravitational waves — cf. e.g. [10]). Making a rotation in the transverse plane by an angle ϑ,

$$\mathbf{e}'_{(1)} = \cos\vartheta\, \mathbf{e}_{(1)} + \sin\vartheta\, \mathbf{e}_{(2)} \ , \quad \mathbf{e}'_{(2)} = -\sin\vartheta\, \mathbf{e}_{(1)} + \cos\vartheta\, \mathbf{e}_{(2)} \tag{13}$$

— which corresponds to $\mathbf{m}' = \mathbf{e}^{-i\vartheta}\mathbf{m}$ — and using Eqs. (6) and (9), we find

$$\mathcal{A}'_+(\tau) = \cos 2\vartheta \mathcal{A}_+ - \sin 2\vartheta \mathcal{A}_\times \ , \quad \mathcal{A}'_\times(\tau) = \sin 2\vartheta \mathcal{A}_+ + \cos 2\vartheta \mathcal{A}_\times \ . \tag{14}$$

Taking $\vartheta = \vartheta_+(\tau) = -\frac{1}{2}\mathrm{Arg}\,\Psi_4$, then $\mathcal{A}'_+ = \frac{1}{2}|\Psi_4|$, $\mathcal{A}'_\times = 0$ — the wave is purely "+" polarized for an observer using $\mathbf{m}_+ = e^{-i\vartheta_+}\mathbf{m}$; if $\vartheta = \vartheta_\times(\tau) = \vartheta_+ + \frac{\pi}{4}$, then $\mathcal{A}'_+ = 0$, $\mathcal{A}'_\times = \frac{1}{2}|\Psi_4|$ — the wave is purely "×" polarized for an observer using $\mathbf{m}_\times = e^{-i\vartheta_\times}\mathbf{m}$. The amplitude $\mathcal{A} = \frac{1}{2}|\Psi_4|$ is invariant under the rotation. A general observer sees a superposition of the two polarization modes shifted by $\frac{\pi}{4}$.

One can similarly show [4], [8] that Ψ_3 and Ψ_2 terms describe longitudinal modes and Coulomb-type effects; Ψ_1 and Ψ_0 terms are equivalent to Ψ_3 and Ψ_4 terms (if $\mathbf{k} \leftrightarrow \mathbf{l}$).

For given principal null vector $\mathbf{k}$ and observer's $\mathbf{u}$ we have chosen the frame vector $\mathbf{e}_{(3)}$ according to Eq. (5), which implies $k_{(1)} = 0 = k_{(2)}$, $k_{(3)} \neq 0$, and makes the physical interpretation based on Eq. (8) simpler. This leads to essentially unique $\mathbf{k}$ and $\mathbf{l}$. More precisely, we easily show the following

Proposition 1: Let $\mathbf{u}$ be the four-velocity ($\mathbf{u}\cdot\mathbf{u}=-1$) and $\mathbf{k}$ be the null vector. Then there exists a unit space-like vector $\mathbf{e}_{(3)}$ which is the projection of the null direction given by $\mathbf{k}$ into the hypersurface orthogonal to $\mathbf{u}$. Such $\mathbf{e}_{(3)}$ is unique (up to reflections $\mathbf{e}_{(3)}\to-\mathbf{e}_{(3)}$) and is given by $\mathbf{e}_{(3)}=-\mathbf{u}+\sqrt{2}\,\mathbf{k}$, where $\mathbf{k}$ satisfies $\mathbf{k}\cdot\mathbf{u}=-1/\sqrt{2}$. Another null vector $\mathbf{l}$ in the plane $(\mathbf{u},\mathbf{e}_{(3)})$ such that $\mathbf{l}\cdot\mathbf{k}=-1$ is then given by $\mathbf{l}\equiv\sqrt{2}\,\mathbf{u}-\mathbf{k}$. The only remaining freedom are rotations in the plane $(\mathbf{e}_{(1)},\mathbf{e}_{(2)})$ perpendicular to $\mathbf{e}_{(3)}$.

In what follows the orthonormal tetrad and the corresponding null frame (5) determined according to Proposition 1 will always be assumed.

Notice that Eqs. (8) represent possible motions seen by an observer with given $\mathbf{u}$. By making the Lorentz boosts to other observers with $\mathbf{u}'$, Ψ_A change (see e.g. [8]). Thus, the "strength of gravitational field" is strongly observer dependent (cf. [2], [12] and point 5. in the discussion following Eq. (34)).

III. THE CHOICE OF COORDINATES AND PARALLELLY PROPAGATED FRAMES

We shall now express our frames in coordinates suitable for spacetimes admitting a *non-twisting* null congruence and give the conditions for the frames to be parallelly transported. The field $\mathbf{k}$ is orthogonal to null hypersurfaces, say $u=$const., so that $k^\mu=g^{\mu\nu}u_{,\nu}$. It is convenient (cf. [9], [13]) to choose as coordinates $u=x^3$, parameter $v=x^0$ along the null geodesics generated by k^μ, and two complex space-like coordinates $\xi=x^1$ and $\bar\xi=x^2$ that label the geodesics on each surface $u=$const. The metric then takes the form

$$g_{\mu\nu}=\begin{pmatrix} 0 & 0 & 0 & g_{03} \\ 0 & g_{11} & g_{12} & g_{13} \\ 0 & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{pmatrix}, \tag{15}$$

where $g_{22}=\overline{g_{11}}$, $g_{23}=\overline{g_{13}}$ since $x^2=\overline{x^1}$; all other components are real, and

$$g_{12}>0\ , \qquad D=g_{12}^2-g_{11}g_{22}>0\ , \tag{16}$$

since the subspace $(\xi,\bar\xi)$ is space-like. The vector $\mathbf{k}$ is simply

$$k^\mu=(k^0,0,0,0)\ , \tag{17}$$

and the four-velocity $\mathbf{u}$ of a particle moving along a geodesic $x^\mu(\tau)$, is given by $u^\mu = (\dot{v}, \dot{\xi}, \dot{\bar{\xi}}, \dot{u})$, where the dot is $d/d\tau$ and $\dot{u} \neq 0$ (otherwise the geodesic would not be time-like).

Proposition 2: In coordinates $(v, \xi, \bar{\xi}, u)$ the interpretation null tetrad introduced in Proposition 1 has the form

$$\begin{aligned}
m^\mu &= \Bigg(\frac{1}{g_{03}\dot{u}} \left[(g_{12}\dot{\xi} + g_{22}\dot{\bar{\xi}} + g_{23}\dot{u}) g_+ - (g_{11}\dot{\xi} + g_{12}\dot{\bar{\xi}} + g_{13}\dot{u}) g_- \exp(-i \mathrm{Arg}\, g_{11}) \right], \\
&\qquad\qquad g_- \exp(-i\mathrm{Arg}\, g_{11}), -g_+, 0 \Bigg) , \\
\bar{m}^\mu &= \Bigg(\frac{1}{g_{03}\dot{u}} \left[(g_{11}\dot{\xi} + g_{12}\dot{\bar{\xi}} + g_{13}\dot{u}) g_+ - (g_{12}\dot{\xi} + g_{22}\dot{\bar{\xi}} + g_{23}\dot{u}) g_- \exp(i \mathrm{Arg}\, g_{11}) \right], \\
&\qquad\qquad -g_+, g_- \exp(i\mathrm{Arg}\, g_{11}), 0 \Bigg) , \qquad (18) \\
l^\mu &= \left(\sqrt{2}\dot{v} + \frac{1}{\sqrt{2}} \frac{1}{g_{03}\dot{u}}, \sqrt{2}\dot{\xi}, \sqrt{2}\dot{\bar{\xi}}, \sqrt{2}\dot{u} \right) , \\
k^\mu &= \left(-\frac{1}{\sqrt{2}} \frac{1}{g_{03}\dot{u}}, 0, 0, 0 \right) ,
\end{aligned}$$

where $g_\pm = \sqrt{(g_{12} \pm \sqrt{D})/(2D)}$. The tetrad is unique up to trivial reflections and rotations $m^\mu \to m^\mu e^{i\vartheta}$. The corresponding orthonormal frame obtained from Eq. (18) using Eq. (5) is

$$\begin{aligned}
e^\mu_{(0)} &= u^\mu = \left(\dot{v}, \dot{\xi}, \dot{\bar{\xi}}, \dot{u} \right) , \\
e^\mu_{(1)} &= \frac{1}{\sqrt{2}} \left(\frac{2}{g_{03}\dot{u}} \mathcal{R}e \left\{ (g_{12}\dot{\xi} + g_{22}\dot{\bar{\xi}} + g_{23}\dot{u})\, G_- \right\}, -\bar{G}_-, -G_-, 0 \right) , \\
e^\mu_{(2)} &= \frac{1}{\sqrt{2}} \left(\frac{2}{g_{03}\dot{u}} \mathcal{I}m \left\{ (g_{12}\dot{\xi} + g_{22}\dot{\bar{\xi}} + g_{23}\dot{u})\, G_+ \right\}, -i\bar{G}_+, iG_+, 0 \right) , \qquad (19) \\
e^\mu_{(3)} &= -\left(\dot{v} + \frac{1}{g_{03}\dot{u}}, \dot{\xi}, \dot{\bar{\xi}}, \dot{u} \right) ,
\end{aligned}$$

where $G_\pm = g_+ \pm g_- \exp(i\mathrm{Arg}\, g_{11})$. The expressions (18) and (19) simplify considerably if $g_{11} = 0$ since in this case $g_- = 0$ and $g_+ = 1/\sqrt{g_{12}} = G_+ = G_-$.

Proof: The last equation in (18) follows from Eq. (17) and $\mathbf{k} \cdot \mathbf{u} = -1/\sqrt{2}$, the equation for l^μ follows from $\mathbf{l} = \sqrt{2}\mathbf{u} - \mathbf{k}$. Vectors $\mathbf{m}$ and $\bar{\mathbf{m}}$ can then be determined from $\mathbf{e}_{\hat{a}} \cdot \mathbf{e}_{\hat{b}} = g_{\mu\nu} e^\mu_{\hat{a}} e^\nu_{\hat{b}} = g_{\hat{a}\hat{b}}$. The conditions $\mathbf{m} \cdot \mathbf{k} = g_{\hat{1}\hat{4}} = 0 = g_{\hat{2}\hat{4}} = \bar{\mathbf{m}} \cdot \mathbf{k}$ imply $m^3 = 0 = \bar{m}^3$, $\mathbf{m} \cdot \mathbf{l} = g_{\hat{1}\hat{3}} = 0$ implies $m^0 = -(l_1 m^1 + l_2 m^2)/l_0$. In given coordinates we have $\bar{m}^1 = \overline{m^2}$

and $\bar{m}^2 = \overline{m^1}$. Denoting $X = m^1$ and $Y = m^2$ we get $m^\mu = (-(l_1 X + l_2 Y)/l_0, X, Y, 0)$ and $\bar{m}^\mu = (-(l_1 \bar{Y} + l_2 \bar{X})/l_0, \bar{Y}, \bar{X}, 0)$. Functions X, Y can be determined as solutions of equations $\mathbf{m} \cdot \mathbf{m} = g_{\hat{1}\hat{1}} = 0 = g_{\hat{2}\hat{2}} = \bar{\mathbf{m}} \cdot \bar{\mathbf{m}}$ and $\mathbf{m} \cdot \bar{\mathbf{m}} = g_{\hat{1}\hat{2}} = 1$,

$$g_{11} X^2 + 2 g_{12} XY + g_{22} Y^2 = 0 \ , \tag{20}$$

$$g_{11} X\bar{Y} + g_{12}(X\bar{X} + Y\bar{Y}) + g_{22} \bar{X} Y = 1 \ . \tag{21}$$

(i) Assume $g_{11} \neq 0$. Then $X \neq 0$, and introducing a complex function C such that $Y = CX$, Eq. (20) implies $C = (-g_{12} \pm \sqrt{D})/g_{22}$, and Eq. (21) gives $X\bar{X} = |X|^2 = (g_{12} \pm \sqrt{D})/(2D)$. Since $X = |X| e^{i\varphi}$, φ being a real function, we have $m^1 = \sqrt{(g_{12} \pm \sqrt{D})/(2D)}\ \exp(i(\vartheta - \mathrm{Arg}\, g_{11}))$, $m^2 = -\sqrt{(g_{12} \mp \sqrt{D})/(2D)}\ \exp(i\vartheta)$, where $\vartheta = \varphi + \mathrm{Arg}\, g_{11}$. The change from the upper to lower signs accompanied by $\vartheta \to -\vartheta + \pi + \mathrm{Arg}\, g_{11}$ results just in $\mathbf{m} \leftrightarrow \bar{\mathbf{m}}$, corresponding to a reflection $\mathbf{e}_{(2)} \leftrightarrow -\mathbf{e}_{(2)}$. By performing rotation $m^\mu \to m'^\mu = e^{-i\vartheta} m^\mu$ we can write the representatives of $\mathbf{m}$ and $\bar{\mathbf{m}}$ given by $\vartheta = 0$ so that we arrive at Eq. (18).
(ii) If $g_{11} = 0$, we simply find $m^1 = 0$ and $m^2 = -1/\sqrt{g_{12}}$. Hence, the null tetrad has the form (18), and this implies the orthonormal frame (19).

In general, the frames $\{\mathbf{e}_a\}$ and $\{\mathbf{e}_{\hat{a}}\}$, related by Eq. (5), are not parallelly transported along the geodesic with tangent $\mathbf{u} = \mathbf{e}_{(0)}$. However, they are if $D\mathbf{k}/d\tau = 0 = D\mathbf{m}/d\tau$. Starting with an arbitrary $\mathbf{m}$, the second condition can always be satisfied by choosing $\mathbf{m}_\parallel = e^{i\vartheta_\parallel} \mathbf{m}$, where $\vartheta_\parallel = i \int_0^\tau \bar{\mathbf{m}} \cdot (D\mathbf{m}/d\tau)\, d\tau + \vartheta_0$, $\vartheta_0 = \mathrm{const}$. We thus arrive at

Proposition 3: Consider a geodesic $x^\mu(\tau) = (v, \xi, \bar{\xi}, u)$ in spacetime with metric (15). Then the orthonormal frame $\{\mathbf{e}_a\}$ given by Eq. (19) and the null tetrad $\{\mathbf{e}_{\hat{a}}\}$ given by Eq. (18) are parallelly transported along the geodesic if

$$g_{12,0}\dot{\xi} + g_{22,0}\dot{\bar{\xi}} + (g_{23,0} - g_{03,2})\dot{u} = 0 \ , \tag{22}$$

and

$$\begin{aligned} \dot{\vartheta}_\parallel(\tau) \ = \ & \frac{i}{2D}\Big[(G_1\bar{G}_1 - G_2\bar{G}_2)\, E \\ & + G_1(2D\dot{m}^1 + m^1(g_{12}\dot{g}_{12} - g_{22}\dot{g}_{11}) + m^2(g_{12}\dot{g}_{22} - g_{22}\dot{g}_{12})) \\ & + G_2(2D\dot{m}^2 + m^1(g_{12}\dot{g}_{11} - g_{11}\dot{g}_{12}) + m^2(g_{12}\dot{g}_{12} - g_{11}\dot{g}_{12}))\Big] \ , \end{aligned} \tag{23}$$

where $G_1 = g_{12}g_- \exp(i\mathrm{Arg}\, g_{11}) - g_{11}g_+$, $G_2 = g_{22}g_- \exp(i\mathrm{Arg}\, g_{11}) - g_{11}g_+$, $E = (g_{12,1} - g_{11,2})\dot{\xi} + (g_{22,1} - g_{12,2})\dot{\bar{\xi}} + (g_{23,1} - g_{13,2})\dot{u} = -\bar{E}$, $m^1 = g_- \exp(-i\mathrm{Arg}\, g_{11})$, and $m^2 = -g_+$. If, in addition, $g_{11} = 0$ then $G_1 = 0$, $G_2 = -\sqrt{g_{12}}$ and Eqs. (22), (23) reduce to

$$g_{12,0}\dot{\xi} + (g_{23,0} - g_{03,2})\dot{u} = 0 \ , \tag{24}$$

and

$$\dot{\vartheta}_{\parallel}(\tau) = -\frac{i}{2}\frac{1}{g_{12}}\Big[g_{12,1}\dot{\xi} - g_{12,2}\dot{\bar{\xi}} + (g_{23,1} - g_{13,2})\dot{u}\Big] \ . \tag{25}$$

Proof: Using $k_\mu k^\mu = 0$ and $k_\mu u^\mu = -1/\sqrt{2}$, it can be shown that $\mathbf{k}$ is parallelly transported if $\Gamma^1_{0\alpha}u^\alpha = 0$. Calculating the Christoffel symbols for the metric (15) we find (22). For proving Eq. (23) we use $m^3 = 0$ and $\bar{m}_0 = 0$, again the condition $\Gamma^1_{0\alpha}u^\alpha = 0$ and other Christoffel symbols for the metric (15).

IV. DEVIATION OF GEODESICS IN THE VACUUM NON-TWISTING TYPE N SPACETIMES WITH COSMOLOGICAL CONSTANT

In this section we apply results given above to the non-twisting type N vacuum spacetimes with non-vanishing Λ. In the preceding paper [1] we showed that all such solutions belong either to the Kundt class of non-expanding gravitational waves which we denoted by symbol $KN(\Lambda)$, or to the Robinson-Trautman class of expanding gravitational waves $RTN(\Lambda, \epsilon)$.

The class $KN(\Lambda)$ can be divided into six invariant canonical subclasses $KN(\Lambda)[\alpha, \beta]$, and the class $RTN(\Lambda, \epsilon)$ into nine invariant canonical subclasses, as analyzed in detail in [1]. All $KN(\Lambda)$ metrics can be written in the form of Eq. (1, I), all $RTN(\Lambda, \epsilon)$ are described by Eq. (19, I). Both classes of metrics are of the form (15) in coordinates $x^\mu = (v, \xi, \bar{\xi}, u)$. In the $KN(\Lambda)$ class we have

$$g_{12} = \frac{1}{p^2} \ , \quad g_{03} = -\frac{q^2}{p^2} \ , \quad g_{33} = F \ , \tag{26}$$

where $p = 1 + \frac{\Lambda}{6}\xi\bar{\xi}$, $q = (1 - \frac{\Lambda}{6}\xi\bar{\xi})\alpha + \bar{\beta}\xi + \beta\bar{\xi}$, $F = \kappa(q^2/p^2)v^2 - ((q^2)_{,u}/p^2)v - (q/p)H$, $\kappa = \frac{\Lambda}{3}\alpha^2 + 2\beta\bar{\beta}$, $H = (f_{,\xi} + \bar{f}_{,\bar{\xi}}) - (\Lambda/3p)(\bar{\xi}f + \xi\bar{f})$; and in the $RTN(\Lambda, \epsilon)$ class

$$g_{12} = v^2 \ , \quad g_{13} = v\bar{\mathrm{A}} \ , \quad g_{23} = v\mathrm{A} \ , \quad g_{03} = \psi \ , \quad g_{33} = 2(\mathrm{A}\bar{\mathrm{A}} + \psi\mathrm{B}) \ , \tag{27}$$

where $\mathrm{A} = \epsilon\xi - vf$, $\mathrm{B} = -\epsilon + \frac{v}{2}(f_{,\xi} + \bar{f}_{,\bar{\xi}}) + \frac{\Lambda}{6}v^2\psi$, $\psi = 1 + \epsilon\xi\bar{\xi}$, $\epsilon = -1, 0, +1$, respectively.

Hence, for the $KN(\Lambda)$ solutions the orthonormal frame (19) is given by

$$
\begin{aligned}
e^{\mu}_{(0)} &= \left(\dot v, \dot\xi, \dot{\bar\xi}, \dot u\right) \ , \\
e^{\mu}_{(1)} &= -\frac{p}{\sqrt{2}}\left(\frac{2}{q^2}\frac{\mathcal{R}e\dot\xi}{\dot u}, 1, 1, 0\right) \ , \\
e^{\mu}_{(2)} &= -\frac{p}{\sqrt{2}}\left(\frac{2}{q^2}\frac{\mathcal{I}m\dot\xi}{\dot u}, i, -i, 0\right) \ , \\
e^{\mu}_{(3)} &= -\left(\dot v - \frac{p^2}{\dot u q^2}, \dot\xi, \dot{\bar\xi}, \dot u\right) \ ,
\end{aligned}
\tag{28}
$$

and for the $RTN(\Lambda, \epsilon)$ solutions we have

$$
\begin{aligned}
e^{\mu}_{(0)} &= \left(\dot v, \dot\xi, \dot{\bar\xi}, \dot u\right) \ , \\
e^{\mu}_{(1)} &= \frac{1}{\sqrt{2}}\frac{1}{v}\left(\frac{2v}{\psi\dot u}\mathcal{R}e\{v\dot\xi + \mathrm{A}\dot u\}, -1, -1, 0\right) \ , \\
e^{\mu}_{(2)} &= \frac{1}{\sqrt{2}}\frac{1}{v}\left(\frac{2v}{\psi\dot u}\mathcal{I}m\{v\dot\xi + \mathrm{A}\dot u\}, -i, i, 0\right) \ , \\
e^{\mu}_{(3)} &= -\left(\dot v + \frac{1}{\psi\dot u}, \dot\xi, \dot{\bar\xi}, \dot u\right) \ .
\end{aligned}
\tag{29}
$$

According to Proposition 3 these frames are parallelly transported along time-like geodesics $x^\mu(\tau) = (v, \xi, \bar\xi, u)$ in the $KN(\Lambda)[\alpha, \beta]$ spacetimes if

$$
\left(\frac{q}{p}\right)_{,\xi} = 0 = \left(\frac{q}{p}\right)_{,\bar\xi} \ , \qquad \dot\vartheta_{\parallel}(\tau) = i\left(\frac{p_{,\xi}}{p}\dot\xi - \frac{p_{,\bar\xi}}{p}\dot{\bar\xi}\right) \ , \tag{30}
$$

and in the case of $RTN(\Lambda, \epsilon)$ solutions if

$$
\dot\xi = f\dot u \ , \qquad \dot{\bar\xi} = \bar f\dot u \ , \qquad \dot\vartheta_{\parallel}(\tau) = \frac{i}{2}\left(f_{,\xi} - \bar f_{,\bar\xi}\right)\dot u \ . \tag{31}
$$

The equation of geodesic deviation is now given by Eq. (8) with $T_{(a)(b)} = 0$. The amplitudes (9) for both classes of spacetimes are calculated in the Appendix. We find that the invariant form of the equation of geodesic deviation with respect to the interpretation frame along any time-like geodesic in the $KN(\Lambda)$ and $RTN(\Lambda, \epsilon)$ spacetimes takes the form

$$
\begin{aligned}
\ddot Z^{(1)} &= \frac{\Lambda}{3}Z^{(1)} - \mathcal{A}_{+}Z^{(1)} + \mathcal{A}_{\times}Z^{(2)} \ , \\
\ddot Z^{(2)} &= \frac{\Lambda}{3}Z^{(2)} + \mathcal{A}_{+}Z^{(2)} + \mathcal{A}_{\times}Z^{(1)} \ , \\
\ddot Z^{(3)} &= \frac{\Lambda}{3}Z^{(3)} \ ,
\end{aligned}
\tag{32}
$$

where the amplitudes of the transverse gravitational wave are given by

$$\mathcal{A}_+(\tau) = \tfrac{1}{2}pq\dot{u}^2\, \mathcal{R}e\,\{f_{,\xi\xi\xi}\}\ , \qquad \mathcal{A}_\times(\tau) = \tfrac{1}{2}pq\dot{u}^2\, \mathcal{I}m\,\{f_{,\xi\xi\xi}\}\ , \tag{33}$$

for the $KN(\Lambda)$ spacetimes, and by

$$\mathcal{A}_+(\tau) = -\tfrac{1}{2}\frac{\psi}{v}\dot{u}^2\, \mathcal{R}e\,\{f_{,\xi\xi\xi}\}\ , \qquad \mathcal{A}_\times(\tau) = -\tfrac{1}{2}\frac{\psi}{v}\dot{u}^2\, \mathcal{I}m\,\{f_{,\xi\xi\xi}\}\ , \tag{34}$$

in the $RTN(\Lambda, \epsilon)$ spacetimes (see Eqs. (45) and (50) in Appendix). Equations (32)-(34) give relative accelerations of the free test particles in terms of their actual positions. They enable us to draw a number of simple conclusions:

1. All particles move isotropically one with respect to the other according to Eqs. (11) if no gravitational wave is present, i.e., if $f_{,\xi\xi\xi} = 0$. In this case both the $KN(\Lambda)$ and $RTN(\Lambda, \epsilon)$ spacetimes are vacuum conformally flat (cf. (43), (48)), and therefore Minkowski ($\Lambda = 0$), de Sitter ($\Lambda > 0$) and anti-de Sitter ($\Lambda < 0$) (see Lemma 1 and 3 in [1]). Such spaces are maximally symmetric, homogeneous, isotropic, and they represent a natural background for other "non-trivial" $KN(\Lambda)$ and $RTN(\Lambda, \epsilon)$ type N solutions.

2. If amplitudes $\mathcal{A}_+$ and $\mathcal{A}_\times$ do not vanish ($f_{,\xi\xi\xi} \neq 0$), the particles are influenced by the wave (see Eq. (12) and subsequent discussion) in a similar way as they are affected by a standard gravitational wave on Minkowski background (cf. [10]). However, if $\Lambda \neq 0$, the influence of the wave adds with the (anti-) de Sitter isotropic expansion (contraction). This makes plausible our interpretation of the $KN(\Lambda)$ and $RTN(\Lambda, \epsilon)$ metrics as *exact gravitational waves propagating on the constant curvature backgrounds*.

3. The wave propagates in the space-like direction of $\mathbf{e}_{(3)}$ and has a *transverse character* since only motions in the perpendicular directions of $\mathbf{e}_{(1)}$ and $\mathbf{e}_{(2)}$ are affected. The propagation direction given by $\mathbf{e}_{(3)}$ coincides with the projection of the Debever-Penrose vector $\mathbf{k}$ on the hypersurface orthogonal to the observer's velocity $\mathbf{u}$ (cf. Proposition 1).

4. There are *two polarization modes* of the wave — "+" and "×", $\mathcal{A}_+$ and $\mathcal{A}_\times$ being the amplitudes. Under rotation (13) in the transverse plane they transform according to Eq. (14) so that the helicity of the wave is 2, as with linearized waves on Minkowski background. For the special choice of the frame given by $\vartheta(\tau) = \vartheta_+ = -\tfrac{1}{2}pq\dot{u}^2 \mathrm{Arg}\,\{f_{,\xi\xi\xi}\}$

for the $KN(\Lambda)$, and by $\vartheta_+ = \frac{1}{4}(\psi/v)\dot{u}^2 \mathrm{Arg}\{f_{,\xi\xi\xi}\}$ for the $RTN(\Lambda,\epsilon)$ spacetimes, the observer views pure "+" polarization, and for $\vartheta_\times = \vartheta_+ + \frac{\pi}{4}$ — pure "×" polarization.

5. The waves have amplitude $\mathcal{A} = \frac{1}{2}pq\dot{u}^2\,|f_{,\xi\xi\xi}|$ for the $KN(\Lambda)$ class and $\mathcal{A} = \frac{1}{2}(\psi/v)\dot{u}^2\,|f_{,\xi\xi\xi}|$ for $RTN(\Lambda,\epsilon)$; this is invariant under rotations (13). However, the amplitude changes under Lorentz transformations to another observer $\mathbf{u}'$ with a spatial velocity $\vec{v} = (v_1, v_2, v_3)$ with respect to the original observer. For type N solutions we get $\mathcal{A}' = (1-v_3)^2/(1-v_1^2-v_2^2-v_3^2)\,\mathcal{A}$. By increasing speed in the wave-propagation direction $\mathbf{e}_{(3)}$ ($v_1 = v_2 = 0, v_3 > 0$), he experiences a weakening of the wave amplitude by factor $(1-v_3)/(1+v_3)$ ($\mathcal{A}' \to 0$ as $v_3 \to 1$) , and by moving in the opposite direction, an increase of the amplitude ($\mathcal{A}' \to \infty$ as $v_3 \to -1$). By increasing speed in the transverse directions $\mathbf{e}_{(1)}$, $\mathbf{e}_{(2)}$, ($v_1^2 + v_2^2 \neq 0, v_3 = 0$), she experiences an increase by the factor $1/(1-v_1^2-v_2^2)$.

In general, all $KN(\Lambda)$ spacetimes contain singularities except for the homogeneous *pp*-waves [8] given by $p = 1 = q$ and $f_{,\xi\xi\xi} = 6c_3(u)$, where $c_3(u)$ is a finite function of u. All other $KN(\Lambda)$ spacetimes are singular at $|\xi| = \infty$ where the amplitudes $\mathcal{A}_+$, $\mathcal{A}_\times$ diverge. Additional singularities in the amplitudes may occur if the coefficients $c_n(u)$, $n \geq 3$, of the analytic expansion of function $f(\xi, u)$ are badly behaved at some u.

$RTN(\Lambda,\epsilon)$ spacetimes also contain singularities. The character of the singularities depends on parameter ϵ and on the form of the function $f(\xi, u)$. As follows from Eq. (34), there is always a singularity at $v = 0$. Another singularity is given by $\psi = \infty$ which occurs only for $\epsilon \neq 0$ at $|\xi| = \infty$. There may be singularities for special forms of f, namely if $f_{,\xi\xi\xi} = \infty$. This occurs at $|\xi| = \infty$ if f contains the terms $c_n\xi^n$, $n \geq 4$. Another type of singularities may appear if some of the coefficients $c_n(u)$ diverge for some values of u. Singularities might be considered as "sources" of waves; however, it is far from certain whether non-singular sources "covering" the regions in which singularities occur can be constructed. The singularities of the $RTN(\Lambda,\epsilon)$ spacetimes can invariantly be characterized by the non-vanishing invariant constructed recently [14] from the second derivatives of the Riemann tensor.

Finally, we shall discuss a special class of geodesics explicitly. Since for $f = f_c = c_0(u) + c_1(u)\xi + c_2(u)\xi^2$ the metrics represent Minkowski, de Sitter or anti-de Sitter space there always exists a transformation of coordinates which brings $g_{\mu\nu}[f = f_c]$ to $g_{\mu\nu}[f = 0]$ (see Lemmas 2 and 4 in [1]). It is thus sufficient to consider only the non-trivial part $f_w \equiv f - f_c$ of function $f(\xi,u)$. Moreover, one can always rearrange its analytic expansion so that $f = \sum_{n=0}^{\infty} c_n(u)\xi^n = \sum_{n=0}^{\infty} \tilde{c}_n(u)(\xi-\xi_0)^n = \sum_{n=3}^{\infty} \tilde{c}_n(u)(\xi-\xi_0)^n + \tilde{f}_c$, ξ_0 being an arbitrary complex constant. Therefore, it is natural to consider structural functions of the form

$$f_w = c_3(u)(\xi-\xi_0)^3 + c_4(u)(\xi-\xi_0)^4 + \dots \,. \tag{35}$$

Consider a *special class of geodesics* characterized by $\xi = \xi_0$ =const. For the $RTN(\Lambda,\epsilon)$ solutions these are geometrically privileged since the interpretation frame (29) is parallelly propagated along them (Eq. (31) is satisfied). One can also find special geodesics for some subclasses of $KN(\Lambda)$: $\xi = \xi_0$ for the PP subclass, $\xi_0 = \pm\sqrt{\frac{6}{\Lambda}}$ for $KN(\Lambda)I$, and $\xi_0 = 0$ for $KN(\Lambda^-)II$. The geodesics $\xi = \xi_0$ have the same forms as geodesics in the "background" since Christoffel symbols for $f = f_w$ and $\xi = \xi_0$ coincide with those for $f = 0$. However, the test particles feel the tidal forces proportional to $\mathcal{A}_+$ and $\mathcal{A}_\times$ given by Eqs. (33), (34). The amplitudes do not vanish since $f_{w,\xi\xi\xi} = 6c_3(u)$ is non-vanishing.

The timelike geodesics $\xi = \xi_0$ =const. in the $RTN(\Lambda>0,\epsilon)[f_w]$ spacetimes are given by

$$v = \frac{\alpha}{1+\epsilon\xi_0\bar{\xi}_0}\left(C_1\cosh\frac{\tau}{\alpha} + C_2\sinh\frac{\tau}{\alpha}\right)\,, \qquad \dot{u} = -\left(C_1\sinh\frac{\tau}{\alpha} + C_2\cosh\frac{\tau}{\alpha} + C_3\right)^{-1}\,, \tag{36}$$

where $\alpha = \sqrt{3/\Lambda}$, C_1, C_2, C_3 are real constants satisfying $C_1^2 - C_2^2 + C_3^2 = 2\epsilon$. The integration of Eq. (36) can be performed explicitly but we do not give it here since only $\dot{u}$ enters the amplitudes. The wave amplitudes (34) are $\mathcal{A}_+ = \mathcal{R}e\,\mathcal{A}$ and $\mathcal{A}_\times = \mathcal{I}m\,\mathcal{A}$ where

$$\begin{aligned}\mathcal{A}(\tau) \;\; &= \;\; -\frac{3}{\alpha}(1+\epsilon\xi_0\bar{\xi}_0)^2\left(C_1\cosh\frac{\tau}{\alpha} + C_2\sinh\frac{\tau}{\alpha}\right)^{-1}\times \\ &\qquad \times\left(C_1\sinh\frac{\tau}{\alpha} + C_2\cosh\frac{\tau}{\alpha} + C_3\right)^{-2}\, c_3(u(\tau))\,. \end{aligned}\tag{37}$$

As proper time τ along geodesics increases, $\tau \to \infty$, particles recede from $v = 0$ and amplitudes decay as $\mathcal{A} \sim \exp(-3\sqrt{\Lambda/3}\,\tau)$, i.e., *waves are damped exponentially.* The spacetime locally approaches the de Sitter universe. This is an explicit demonstration of the *cosmic no-hair conjecture* (see, e.g. [5]) under the presence of waves within exact model spacetimes. (For cosmic no-hair conjecture in the Robinson-Trautman spacetimes of Petrov type II see [6], [7].)

Similarly, for the $KN(\Lambda>0)I[f_w]$ subclass (representing the only spacetimes of the $KN(\Lambda)$ type admitting $\Lambda > 0$) the geodesics $\xi = \xi_0 = \pm\sqrt{6/\Lambda}$ are given by

$$v = C_1\exp\left(\frac{\tau}{\alpha}\right)\,, \qquad u = -\frac{1}{2C_1}\exp\left(-\frac{\tau}{\alpha}\right) + C_2\,, \tag{38}$$

and

$$v = C_1 \sinh\left(\frac{\tau}{\alpha} + 2\tau_0\right) \ , \qquad u = \frac{1}{2C_1} \tanh\left(\frac{\tau}{2\alpha} + \tau_0\right) + C_2 \ , \tag{39}$$

with C_1, C_2, τ_0 constants. For observers moving along these geodesics,

$$\mathcal{A}(\tau) = \pm 12\sqrt{\frac{6}{\Lambda}}\dot{u}^2(\tau)\, c_3(u(\tau)) \ . \tag{40}$$

After substitution of the explicit dependence of $u(\tau)$ we see that as $\tau \to +\infty$ the amplitudes behave like $\mathcal{A} \sim \exp(-2\sqrt{\Lambda/3}\,\tau)$. Again, gravitational waves are damped exponentially and the cosmic no-hair conjecture is confirmed.

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APPENDIX: GRAVITATIONAL WAVE AMPLITUDES

We calculate amplitudes $\mathcal{A}_+ = \frac{1}{2}\mathcal{R}e\Psi_4$ and $\mathcal{A}_\times = \frac{1}{2}\mathcal{I}m\Psi_4$ by using differential forms. Let $\{\mathbf{e}_{\hat{a}}\} = \{\mathbf{m}, \bar{\mathbf{m}}, \mathbf{l}, \mathbf{k}\}$ be a null tetrad, $\mathbf{m} = \mathbf{e}_{\hat{1}} = m^\mu\partial_\mu$, $\bar{\mathbf{m}} = \mathbf{e}_{\hat{2}} = \bar{m}^\mu\partial_\mu$, $\mathbf{l} = \mathbf{e}_{\hat{3}} = l^\mu\partial_\mu$, $\mathbf{k} = \mathbf{e}_{\hat{4}} = k^\mu\partial_\mu$. The dual basis $\{\boldsymbol{\omega}^{\hat{a}}\}$ is given by one-forms $\boldsymbol{\omega}^{\hat{1}} = \bar{m}_\mu dx^\mu$, $\boldsymbol{\omega}^{\hat{2}} = m_\mu dx^\mu$, $\boldsymbol{\omega}^{\hat{3}} = -k_\mu dx^\mu$, $\boldsymbol{\omega}^{\hat{4}} = -l_\mu dx^\mu$; the metric is $ds^2 = g_{\hat{a}\hat{b}}\, \boldsymbol{\omega}^{\hat{a}}\boldsymbol{\omega}^{\hat{b}} = 2\,\boldsymbol{\omega}^{\hat{1}}\boldsymbol{\omega}^{\hat{2}} - 2\,\boldsymbol{\omega}^{\hat{3}}\boldsymbol{\omega}^{\hat{4}}$ with $g_{\hat{1}\hat{2}} = \mathbf{e}_{\hat{1}}\cdot\mathbf{e}_{\hat{2}} = 1$ and $g_{\hat{3}\hat{4}} = \mathbf{e}_{\hat{3}}\cdot\mathbf{e}_{\hat{4}} = -1$. The natural choice of the null basis for the metric $KN(\Lambda)$ is

$$\boldsymbol{\omega}^{\hat{1}} = \frac{d\bar{\xi}}{p} \ , \quad \boldsymbol{\omega}^{\hat{2}} = \frac{d\xi}{p} \ , \quad \boldsymbol{\omega}^{\hat{3}} = \frac{q^2}{p^2}du \ , \quad \boldsymbol{\omega}^{\hat{4}} = dv - \frac{1}{2}\frac{p^2}{q^2}F du \ ; \tag{41}$$

in coordinates $x^\mu = (v, \xi, \bar{\xi}, u)$ we have

$$\begin{aligned} m^\mu &= (0,0,p,0) \ , \qquad & \bar{m}^\mu &= (0,p,0,0) \ , \\ k^\mu &= (1,0,0,0) \ , \qquad & l^\mu &= (\frac{1}{2}\frac{p^4}{q^4}F, 0, 0, \frac{p^2}{q^2}) \ . \end{aligned} \tag{42}$$

The non-vanishing components of the Weyl tensor in this null tetrad are

$$C_{\hat{3}\hat{2}\hat{3}\hat{2}} \equiv \Psi_4 = \frac{1}{2}\frac{p^5}{q^3}f_{,\xi\xi\xi} = \overline{C_{\hat{3}\hat{1}\hat{3}\hat{1}}} \ . \tag{43}$$

The interpretation null tetrad for the $KN(\Lambda)$ spacetimes, given by Eq. (18), reads

$$
\begin{aligned}
m^\mu &= \left(-\frac{p}{q^2}\frac{\dot{\xi}}{\dot{u}}, 0, -p, 0 \right) , \qquad & \bar{m}^\mu &= \left(-\frac{p}{q^2}\frac{\dot{\bar{\xi}}}{\dot{u}}, -p, 0, 0 \right) , \\
k^\mu &= \left(\frac{1}{\sqrt{2}\dot{u}}\frac{p^2}{q^2}, 0, 0, 0 \right) , \qquad & l^\mu &= \left(\sqrt{2}\dot{v} - \frac{1}{\sqrt{2}\dot{u}}\frac{p^2}{q^2}, \sqrt{2}\dot{\xi}, \sqrt{2}\dot{\bar{\xi}}, \sqrt{2}\dot{u} \right) .
\end{aligned}
\tag{44}
$$

The relation between tetrads (42) and (44) is given by the Lorentz transformation, $\mathbf{k}_{natur} = A\mathbf{k}_{interp}$, $\mathbf{l}_{natur} = (\mathbf{l}_{interp} + Be^{i\vartheta}\bar{\mathbf{m}}_{interp} + \bar{B}e^{-i\vartheta}\mathbf{m}_{interp} + B\bar{B}\,\mathbf{k}_{interp})/A$, $\mathbf{m}_{natur} = e^{-i\vartheta}\mathbf{m}_{interp} + B\,\mathbf{k}_{interp}$, where $A = \sqrt{2}\dot{u}\,q^2/p^2$, $B = -\sqrt{2}\,\dot{\xi}/p$, $\vartheta = \pi$. The coefficients Ψ_A transform (see [8])

$$
\begin{aligned}
\Psi_4^{interp} &= A^2\Psi_4^{natur} = pq\dot{u}^2 f_{,\xi\xi\xi} , \\
\Psi_3^{interp} &= \Psi_2^{interp} = \Psi_1^{interp} = \Psi_0^{interp} = 0 .
\end{aligned}
\tag{45}
$$

Similarly, the natural choice of null basis for the $RTN(\Lambda, \epsilon)$ metric is

$$
\boldsymbol{\omega}^{\hat{1}} = vd\bar{\xi} + \bar{\mathrm{A}}du , \quad \boldsymbol{\omega}^{\hat{2}} = vd\xi + \mathrm{A}du , \quad \boldsymbol{\omega}^{\hat{3}} = \psi du , \quad \boldsymbol{\omega}^{\hat{4}} = -dv - \mathrm{B}du , \tag{46}
$$

so that

$$
\begin{aligned}
m^\mu &= (0, 0, \frac{1}{v}, 0) , \qquad & \bar{m}^\mu &= (0, \frac{1}{v}, 0, 0) , \\
k^\mu &= (-1, 0, 0, 0) , \qquad & l^\mu &= (-\frac{\mathrm{B}}{\psi}, -\frac{\mathrm{A}}{v\psi}, -\frac{\bar{\mathrm{A}}}{v\psi}, \frac{1}{\psi}) .
\end{aligned}
\tag{47}
$$

For this tetrad we obtain non-vanishing components

$$
C_{\hat{3}\hat{2}\hat{3}\hat{2}} \equiv \Psi_4 = -\frac{1}{2v\psi}\, f_{,\xi\xi\xi} = \overline{C_{\hat{3}\hat{1}\hat{3}\hat{1}}} . \tag{48}
$$

The interpretation null tetrad (18) reads

$$
\begin{aligned}
m^\mu &= \left(\frac{1}{\psi\dot{u}}(v\dot{\xi} + \mathrm{A}\dot{u}), 0, -\frac{1}{v}, 0 \right) , \qquad & \bar{m}^\mu &= \left(\frac{1}{\psi\dot{u}}(v\dot{\bar{\xi}} + \bar{\mathrm{A}}\dot{u}), -\frac{1}{v}, 0, 0 \right) , \\
k^\mu &= \left(-\frac{1}{\sqrt{2}}\frac{1}{\psi\dot{u}}, 0, 0, 0 \right) , \qquad & l^\mu &= \left(\sqrt{2}\dot{v} + \frac{1}{\sqrt{2}}\frac{1}{\psi\dot{u}}, \sqrt{2}\dot{\xi}, \sqrt{2}\dot{\bar{\xi}}, \sqrt{2}\dot{u} \right) .
\end{aligned}
\tag{49}
$$

The relation between the tetrads (47) and (49) is again given by the Lorentz transformation with $A = \sqrt{2}\,\dot{u}\psi$, $B = -\sqrt{2}\,(v\dot{\xi} + \mathrm{A}\dot{u})$, $\vartheta = \pi$. We thus get

$$
\begin{aligned}
\Psi_4^{interp} &= A^2\Psi_4^{natur} = -\frac{\psi}{v}\dot{u}^2\, f_{,\xi\xi\xi} , \\
\Psi_3^{interp} &= \Psi_2^{interp} = \Psi_1^{interp} = \Psi_0^{interp} = 0 .
\end{aligned}
\tag{50}
$$

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